

# Role of Tilted Congruence and $f(R)$ Gravity on Regular Compact Objects

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## Abstract

The purpose of this paper is to check the impact of observer and Palatini  $f(R)$  terms in the formulations of inhomogeneity factors of spherical relativistic systems. We consider Lemaître-Tolman-Bondi dynamical model as a compact object and studied its evolution with both tilted and non-tilted observers. We performed our analysis for particular cases of fluid distribution in tilted frame and found some energy density irregularity variables. We found that these variables are drastically different from those observed by non-tilted observer. The conformal flat dust and perfect matter contents are homogeneous as long as they impregnate vacuum core. However, this restriction is relaxed, when the complexity in the fluid description is increased. The radial fluid velocity due to tilted congruences and Palatini  $f(R)$  curvature terms tend to produce hindrances in the appearance of energy-density inhomogeneities in the initially regular spherical stellar populations.

**Keywords:**  $f(R)$  gravity; Structure scalars; Relativistic dissipative fluids.

**PACS:** 04.20.Cv; 04.40.Nr; 04.50.Kd.

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# 1 Introduction

The modern cosmology is inferred as the study of geometry and matter in the universe which leads to new theoretical ideas about the theories of gravity analogous with current observations. In 1920s, it was Friedmann and Lemaître who introduced the concept of expanding universe which gained significance due to Hubble's observations in 1930s. We are still uncertain about the dark side of the universe, which includes dark energy (DE) and dark matter (DM), but have confidence that it has some crucial role in astrophysics and cosmology. The demonstration of the current accelerated phase of the evolutionary universe proved the dominance of DE with immense negative pressure [1]. An arguable alternative to DE is the generalization of general relativity (GR) which can render cosmic acceleration (for reviews on not only dark energy problem but also modified gravity theories, see, for example, [2]). A simple possible generalization to GR is the inclusion of non-generic function of Ricci scalar in the Einstein-Hilber action which can describe the accelerated expansion and termed as  $f(R)$  theory. In deriving the modified field equations, the procedure which involve the metric and connections to be independent while performing the variations in the action is termed as *Palatini approach*. There exists distinct  $f(R)$  models that meet local and cosmological constraints and can be found in literature [3]. The comprehensive study on the effectiveness and viability of Palatini approach in  $f(R)$  theory of gravity as compared to observational solar system data has been presented by Olmo with his coworkers [4].

Li and Chu [5] presented a framework to study the late-time cosmic acceleration by constraining the  $f(R)$  correction under Palatini version to GR with high red-shift parameter. Kainulainen *et al.* [6] analyzed the exterior and interior geometries of stars by exploring Tolman-Oppenheimer-Volkoff equations in the background of both Palatini and metric  $f(R)$  theory. Fay *et al.* [7] provided a systematic study for different  $f(R)$  gravity models discussing the cosmological dynamics in Palatini version. Shojai and Shojai [8] studied the features of geodesic deviation and its congruences by making use of Raychaudhuri's equation in Palatini  $f(R)$  gravity. Sotiriou and Faraoni [9] surveyed all versions of  $f(R)$  gravity from literature and presented their most significant views comprehensively. Kucukakca and Camci [10] explored exact solutions with flat Friedmann-Robertson-Walker (FRW) model for cosmic scale factor by adopting Noether gauge symmetry approach under Palatini  $f(R)$  formalism.

It was enlightened that the universe is not isotropic and homogeneous at the galactic epochs. To understand the dynamics of anisotropic and inhomogeneous universe, several cosmological models have been proposed. Penrose and Hawking [11] discussed the irregularities density distribution of spherical relativistic stars by means of Weyl invariant. Energy density inhomogeneities also bring forward a crucial role in the process of gravitational collapse which may lead to appearance of naked singularity [12]. However, the exact relation between the final outcome of the collapse and density inhomogeneities is still unidentified.

During the evolution of self-gravitating relativistic models, the role of energy density inhomogeneity have gained much significance [13]. Herrera *et al.* [14] discussed the role of density inhomogeneities on the evolution and structure formation of spherical anisotropic objects. Recently, the role of Weyl tensor and super-Poynting vector in various aspects of dissipative and non-dissipative self-gravitating fluids have been a subject of keen interest [15].

Mena *et al.* [16] explored the role of inhomogeneity and anisotropy for a spherically symmetric dust cloud. Di Prisco *et al.* [17] looked into non-adiabatic spherically symmetric collapsing process and explored the role of energy density inhomogeneities. Chuang *et al.* [18] explored the possibilities of emergence of inhomogeneities for acceleratory expanding universe. Herrera *et al.* [19] studied the dynamics of dissipative spherical collapse and demonstrated a relation between density inhomogeneities and Weyl tensor. Herrera [20] formulated inhomogeneity factors for adiabatic and non-adiabatic relativistic matters and claimed that the system must satisfy these constraints to achieve stable configurations. Bhatti and his coworkers [21] examined the impact of extra Ricci curvature terms on the stability of spherical compact objects filled with anisotropic relativistic matter distributions. Yousaf *et al.* [22] explored the contribution of  $f(R, T)$  extra curvature terms in the outcomes of inhomogeneity factors for dissipative spherical system. They also found factors that causes the maintenance of homogeneous or inhomogeneous matter state, when the system departs hydrostatic equilibrium phase in modified gravity [23].

A system is said to be tilted if its fluid four-velocity and group of orbits are not orthogonal and non-tilted otherwise. It is established in literature that some new interesting results can be achieved due to tilted observer. The general tilted dynamics of cosmological models have been considered by Ellis and his collaborators [24] as well as Bali and his collaborators [25]. The initial attempt to examine tilted models qualitatively has been made for Bianchi type II cosmological models [26]. Pawar *et al.* [27] studied tilted plane symmetric cosmological models of dissipating isotropic fluid and investigated that the resulting universe is shearing, expanding and non-rotating. Apostolopoulos [28] interpreted the dynamical and geometric features for one class of Bianchi models by presenting the evolution equations and equilibrium points in tilted and non-tilted frames. Sahu and Kumar [29] explored the exact solutions for tilted Bianchi-I cosmological model and examined their geometrical and physical properties. Sharif and Bhatti [30] explored the tilted compact objects and developed relationships between tilted and non-tilted variables which are used in analyzing different physical quantities. The influences of extra curvature terms coming from modified gravity on the formulation of inhomogeneity factors [31] and evolution of stellar collapse [32] have also been analyzed.

Through the present paper, we explore the inhomogeneity factors which can control an initially homogeneous system with the evolution of time. The format of this paper is outlined as follows. In the next section, we construct all the basic equations by introducing the concept

of tilted observer in the framework of Palatini  $f(R)$  theory. In section **3**, the kinematical quantities, dynamical as well as evolution equations are explored from the congruence of tilted observer. Section **4** is devoted to characterize the inhomogeneity factors with some particular constraints on the matter profile. The last section concludes our main findings.

## 2 Palatini $f(R)$ Formalism

The modified gravity theories could be considered as a powerful tool to understand the enigmatic cosmic evolution. For Palatini  $f(R)$  gravity, the Einstein-Hilbert action is modified as [33]

$$S_{f(R)} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + \mathcal{S}_M, \quad (1)$$

where  $\kappa$  and  $\mathcal{S}_M$  are constant number with appropriate dimensions, for instance  $\kappa = 8\pi G$  for GR action and matter fields action, respectively, while  $R \equiv g^{\gamma\delta} R_{\gamma\delta}$ ,  $R_{\gamma\delta} \equiv R^\mu_{\gamma\mu\delta}$  with  $R^\mu_{\nu\gamma\delta} = \partial_\gamma \Gamma^\mu_{\delta\nu} - \partial_\delta \Gamma^\mu_{\gamma\nu} + \Gamma^\mu_{\gamma\sigma} \Gamma^\sigma_{\delta\nu} - \Gamma^\mu_{\delta\sigma} \Gamma^\sigma_{\gamma\nu}$  indicates Riemann tensor components, the field intensity due to connections  $\Gamma^\mu_{\gamma\nu}$ . As the connection is found dynamically, therefore one cannot consider  $\Gamma^\mu_{\gamma\delta} = \Gamma^\mu_{\delta\gamma}$ . Due to this reason, we shall keep  $\Gamma^\mu_{\gamma\delta} \neq \Gamma^\mu_{\delta\gamma}$  along with  $g_{\gamma\delta} = g_{\delta\gamma}$  in our variations. Varying the above action with  $g_{\gamma\delta}$  and  $\Gamma^\mu_{\gamma\delta}$  provide

$$\delta S_{f(R)} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ \left( f_R R_{(\gamma\delta)} - \frac{1}{2} g_{\gamma\delta} f \right) \delta g^{\gamma\delta} + g^{\gamma\delta} f_R \delta R_{\gamma\delta} \right] + \delta \mathcal{S}_M, \quad (2)$$

where  $R_{(\gamma\delta)}$  and  $f_R$  indicate symmetric component of the Ricci tensor and partial Ricci scalar derivation of  $f$ , respectively. The variations of  $R_{\gamma\delta}$  can be expressed as

$$\delta R_{\gamma\delta} = \nabla_\sigma (\delta \Gamma^\sigma_{\delta\gamma}) - \nabla_\delta (\delta \Gamma^\sigma_{\sigma\gamma}) + 2\mathcal{S}^\sigma_{\mu\delta} \delta \Gamma^\mu_{\sigma\gamma}, \quad (3)$$

where  $\mathcal{S}^\sigma_{\mu\delta}$  is the torsion tensor defined as  $\mathcal{S}^\sigma_{\mu\delta} = \frac{1}{2}(\Gamma^\sigma_{\mu\delta} - \Gamma^\sigma_{\delta\mu})$ . The role of  $\delta R_{\gamma\delta}$  quantity in the action (1) can be given as

$$\begin{aligned} \int d^4x \sqrt{-g} g^{\gamma\delta} \delta R_{\gamma\delta} &= \int d^4x [\nabla_\sigma (\sqrt{-g} \mathcal{P}^\sigma) + \delta \Gamma^\sigma_{\delta\gamma} \{2\sqrt{-g} g^{\gamma\mu} \mathcal{S}^\delta_{\sigma\mu} \\ &\quad + \nabla_\lambda (\sqrt{-g} g^{\gamma\lambda} f_R) - \nabla_\sigma (\sqrt{-g} g^{\gamma\delta} f_R)\}], \end{aligned} \quad (4)$$

where  $\mathcal{P}^\sigma = (g^{\gamma\delta} \delta \Gamma^\sigma_{\gamma\delta} - g^{\gamma\sigma} \delta \Gamma^\rho_{\rho\gamma}) f_R$ . The first term in the above equation takes the following form

$$\nabla_\sigma (\sqrt{-g} \mathcal{P}^\sigma) = \partial_\sigma (\sqrt{-g} \mathcal{P}^\sigma) + \sqrt{-g} f_R [g^{\gamma\delta} S^\lambda_{\lambda\sigma} - \delta^\delta_{\sigma} g^{\gamma\mu} \mathcal{S}^\rho_{\rho\mu}] \delta \Gamma^\sigma_{\delta\gamma}. \quad (5)$$

Using these values along with the identity of the surface quantity at the hypersurfaces, i.e.,  $\int d^4x(\sqrt{-g}\mathcal{P}^\sigma) = 0$  in Eq.(4), the field equations can be established as

$$f_R(R)R_{(\gamma\delta)} - [g_{\gamma\delta}f(R)]/2 = \kappa T_{\gamma\delta}, \quad (6)$$

$$-\nabla_\sigma(\sqrt{-g}g^{\gamma\delta}f_R) + \delta^\delta_\sigma \nabla_\lambda(\sqrt{-g}g^{\gamma\lambda}f_R) + 2\sqrt{-g}f_R(g^{\gamma\delta}\mathcal{S}^\rho_{\rho\sigma} - \delta^\delta_\sigma g^{\gamma\lambda}\mathcal{S}^\mu_{\mu\lambda} + g^{\gamma\mu}\mathcal{S}^\delta_{\sigma\mu}) = \mathcal{H}^{\delta\gamma}_\sigma, \quad (7)$$

where  $\mathcal{H}^{\delta\gamma}_\sigma = -(\delta\mathcal{S}_M/\delta\Gamma^\sigma_{\delta\gamma})$  and  $T_{\gamma\delta} = -(\delta\mathcal{S}_M/\delta g^{\gamma\delta})(2/\sqrt{-g})$ . Since we have considered that the fluid content is not coupled with connection, therefore  $\mathcal{H}^{\delta\gamma}_\sigma = 0$ . To have a torsionless background, we need to impose  $\mathcal{S}^\delta_{\sigma\rho} = 0$ . In this context, Eq.(7) turns out to be

$$\nabla_\mu(g^{\gamma\delta}\sqrt{-g}f_R(R)) = 0, \quad (8)$$

One can also obtain the similar configurations as mentioned above by removing the torsionless condition (for details please see [34]). On solving Eq.(8) (without imposing torsionless condition, i.e., for the sake of general discussion), we found the relation of connection as follows

$$\Gamma^\mu_{\gamma\delta} = \mathcal{C}^\mu_{\gamma\delta} - \frac{2}{3}\mathcal{A}_\gamma\delta^\mu_\delta, \quad (9)$$

where

$$\mathcal{C}^\mu_{\gamma\delta} = \frac{1}{2}h^{\mu\sigma}(\partial_\gamma h_{\sigma\delta} + \partial_\delta h_{\sigma\gamma} - \partial_\sigma h_{\gamma\delta}), \quad \text{with } h_{\gamma\delta} = f_R g_{\gamma\delta} \quad (10)$$

and  $\mathcal{A}_\mu \equiv S^\gamma_{\gamma\mu}$ . Equation (9) has expressed the connection by means of matter, metric and  $\mathcal{A}_\mu$ . For torsion-less environment, the quantity  $\mathcal{A}_\mu$  will be zero. Substituting Eq.(9) in Eq.(8), it follows that

$$\begin{aligned} & \frac{1}{f_R}(\nabla_\gamma \nabla_\delta - g_{\gamma\delta}\Box)f_R + \frac{1}{2}g_{\gamma\delta}R + \frac{\kappa}{f_R}T_{\gamma\delta} + \frac{1}{2}g_{\gamma\delta}\left(\frac{f}{f_R} - R\right) \\ & + \frac{3}{2f_R^2}\left[\frac{1}{2}g_{\gamma\delta}(\nabla f_R)^2 - \nabla_\gamma f_R \nabla_\delta f_R\right] - \hat{R}_{\gamma\delta} = 0, \end{aligned} \quad (11)$$

where  $R = R(g)$ ,  $R_{\gamma\delta} = R_{\gamma\delta}(g)$  and  $\nabla_\gamma \nabla_\delta f_R$  are calculated through Levi-Civita connection of the usual metric  $g_{\gamma\delta}$ . The trace of the above equation can be expressed as

$$Rf_R(R) - 2f(R) = \kappa T, \quad (12)$$

where  $T \equiv g^{\gamma\delta}T_{\gamma\delta}$  is the trace of usual energy momentum tensor. Equation (12) has expressed Palatini curvature scalar by means of  $T$  thereby indicating  $R$  and  $f_R$  as the functions of  $T$ , i.e.,  $R = R(T)$  and  $f_R = f_R(T)$ . This has made their dependence on metric variables, not on independent connections. The vacuum case, i.e.,  $T_{\gamma\delta} = 0$  would necessarily leads the differential equation to has a constant solution that would secure connections to be well-known Levi-Civita. Further, this would also assign constant value to  $f_R$ . The Palatini equation of motion (11) can be manipulated as

$$G_{\gamma\delta} = \frac{\kappa}{f_R}(T_{\gamma\delta} + \mathcal{T}_{\gamma\delta}), \quad (13)$$

where

$$\begin{aligned} \mathcal{T}_{\gamma\delta} = & \frac{1}{\kappa}(\nabla_\gamma \nabla_\delta - g_{\gamma\delta} \square) f_R - \frac{f_R}{2\kappa} g_{\gamma\delta} \left( R - \frac{f}{f_R} \right) \\ & + \frac{3}{2\kappa f_R} \left[ \frac{1}{2} g_{\gamma\delta} (\nabla f_R)^2 - \nabla_\gamma f_R \nabla_\delta f_R \right], \end{aligned}$$

while  $G_{\gamma\delta} \equiv R_{\gamma\delta} - \frac{1}{2}g_{\gamma\delta}R$  is the Einstein tensor,  $\square = \nabla_\gamma \nabla_\delta g^{\gamma\delta}$  is a de Alembert operator.

The most general mathematical expression for Lemaître-Tolman-Bondi (LTB) spacetime is [35]

$$ds_-^2 = dt^2 - \frac{A'^2}{(w+v)} dr^2 - C^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (14)$$

where  $v$  could be 0 or  $\pm 1$ ,  $w = w(r)$  following the constraint  $w + v \geq 0$  and prime indicates  $\frac{\partial}{\partial r}$  operator. This spacetime has been used to study many burning and useful phenomena of our anisotropic and inhomogeneous universe. It is worthy to mention that on orders much shorter than Hubble radius, our universe mass density could be predicted as homogeneous, however, this density regularity is non longer exists at all scales. One can consider this to be an applicable scenario for distances larger than 100 *Mpc*. The galactic population has been appeared to be spatially inhomogeneous for  $r$  less than 100 *Mpc/h*. There has been interesting literature on this issue [36]. Without loss of generality, one can take  $B = A'$  along with  $w + v = 1$ . Under this background, the non-static diagonal irrotational LTB metric is found as follows

$$ds_-^2 = -dt^2 + B^2 dr^2 + C^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (15)$$

The geometry of any relativistic celestial bodies is designed by the gravitational effects coming from its matter source. Such sources are peculiarly connected with their four-velocities, thus presenting fluid four-velocities as prominent factors in the formulation of energy-momentum tensors. The illustrations as well as congruence kinematics of the gravitational sources could be dissimilar, if the two feasible relativistic explanations of a given

spacetime are linked through the boost of one of the observer congruences regarding to the other one. For instance, FRW (with zero curvature) is a solution of field equation that is coupled with two different relativistic matter distributions. The first one is ideal fluid and second is viscous radiating matter source, depending upon the choice of four-velocity. The former is the solution for those rest observers who is configuring with reference to time-like congruence, developed by eigenvectors of  $R_{\gamma\delta}$ , while the observer who is moving with relative velocity regarding the first previous frame will see this to be solution of the later fluid source. Based upon this concept, we first suppose the comoving coordinate frame, under which the non-interacting particles have the four-velocity

$$u^\gamma = (1, 0, 0, 0), \quad (16)$$

with the stress-energy tensor

$$T_{\gamma\delta} = \hat{\rho} u_\gamma u_\delta, \quad (17)$$

where  $\hat{\rho}$  is the energy density. To get tilted congruence, we assume that fluid distribution has some velocity  $\omega = \omega(r)$  with respect to a new reference frame. Now, we apply Lorentz boost from locally Minkowskian frame carrying dust particles to this new frame. Consequently, this gives rise to the concept of tilted congruences, supported by the following four-vector field

$$U^\gamma = \left( \frac{1}{\sqrt{1-\omega^2}}, \frac{\omega}{B\sqrt{1-\omega^2}}, 0, 0 \right). \quad (18)$$

The fluid corresponding to the tilted frame and vector field  $U_\gamma$  is the radiating anisotropic matter distribution, with the energy-momentum tensor

$$T_{\gamma\delta} = (\rho + P_\perp) U_\gamma U_\delta + \epsilon l_\gamma l_\delta - P_\perp g_{\gamma\delta} + q_\gamma U_\delta + (P_r - P_\perp) S_\gamma S_\delta + q_\delta U_\gamma, \quad (19)$$

where  $\rho$ ,  $q_\gamma$ ,  $\epsilon$ ,  $P_\perp$  and  $P_r$  are energy density, heat flux vector, radiation density, tangential and radial pressures, respectively. The quantities  $S^\gamma$  and  $l^\gamma$  are four-vectors with definitions

$$S^\gamma = \left( \frac{\omega}{\sqrt{1-\omega^2}}, \frac{1}{B\sqrt{1-\omega^2}}, 0, 0 \right), \quad l^\gamma = \left( \frac{1+\omega}{\sqrt{1-\omega^2}}, \frac{1+\omega}{B\sqrt{1-\omega^2}}, 0, 0 \right). \quad (20)$$

The heat flux scalar can be obtained through  $S^\gamma$  as

$$q^\gamma = q S^\gamma. \quad (21)$$

All 4-vectors associated with tilted congruences are satisfying

$$U^\gamma U_\gamma = -1 = l_\gamma U^\gamma, \quad S^\gamma S_\gamma = 1 = l_\gamma S^\gamma, \quad l^\gamma l_\gamma = 0 = S^\gamma U_\gamma = U^\gamma q_\gamma.$$

For tilted-congruences, we would take  $f(R) = R + \frac{\delta^4}{R}$ , with  $\delta > 0$  [37].

### 3 Palatini $f(R)$ Ellis Equations

This section is devoted to develop relationship between Weyl scalar and LTB dynamical variables widely known as Ellis equations with á lá Palatini formalism. For this purpose, we will formulate some equations with the help of mass function and field equations. The Palatini  $f(R)$  equations of motion provide the energy variation of the stellar population gradients respecting time and proximate surfaces. Through contracted Bianchi identities

$$Y_{\delta;\gamma}^\gamma = 0, \quad \text{with} \quad Y_\delta^\gamma = T_\delta^\gamma + \mathcal{T}_\delta^\gamma,$$

and  $f(R, T)$  field equations with tilted congruence background can be found as

$$\begin{aligned} & \tilde{\rho}^* + \tilde{\rho}\Theta + \tilde{q}^\dagger + \tilde{q} \left\{ \omega\Theta + \frac{\sqrt{1-\omega^2}}{B} \left( \frac{2C'}{C} + \frac{f'_R}{f_R} \right) + \frac{2\dot{\omega}}{\sqrt{1-\omega^2}} \right\} + \frac{\tilde{\rho}f_R^*}{f_R} \\ & + \frac{\tilde{q}f_R^\dagger}{f_R} + \frac{\omega P'_\perp}{B} + P_\perp \left( \Theta + \frac{\dot{f}_R}{f_R} + \frac{\omega f'_R}{f_R} \right) + \mathcal{D}_0 = 0, \end{aligned} \quad (22)$$

$$\begin{aligned} & \tilde{P}_r^\dagger + a(\tilde{\rho} + \tilde{P}_r) + \frac{2\tilde{q}}{3} [2\Theta + \sigma - 3\omega(\ln C)^\dagger] + \tilde{q}^* + \frac{\omega f_R^*}{f_R} (\tilde{\rho} + P_\perp) - \tilde{q}\sqrt{1-\omega^2} \\ & \times \left( \frac{\dot{B}}{B} + \frac{2\dot{C}}{C} \right) + \frac{1}{f_R\sqrt{1-\omega^2}} \left( \tilde{q}\omega^2 \dot{f}_R - \frac{\tilde{\rho}f'_R}{B} - \frac{P_\perp f'_R}{B} \right) - \sqrt{1-\omega^2} (P_\perp \dot{\omega}) \\ & - \frac{\omega^2 P'_\perp}{B\sqrt{1-\omega^2}} + \frac{\omega}{\sqrt{1-\omega^2}} [\dot{\mu} + (\omega\tilde{q})] + \mathcal{D}_1 = 0, \end{aligned} \quad (23)$$

where  $\Theta$  is a Palatini  $f(R)$  expansion scalar,  $\sigma$  is a shearing quantity related to shear tensor. Their values for LTB spacetime are

$$\Theta = \frac{1}{\sqrt[3]{1-\omega^2}} \left[ \omega\dot{\omega} + \frac{\omega'}{B} + (1-\omega^2) \left\{ \frac{\dot{B}}{B} + \frac{2\dot{f}_R}{f_R} + \frac{2\dot{C}}{C} + \frac{2\omega C'}{CB} + \frac{\omega f'_R}{Bf_R} \right\} \right], \quad (24)$$

$$\sigma = \frac{1}{\sqrt[3]{1-\omega^2}} \left[ \omega\dot{\omega} + \frac{\omega'}{B} + (1-\omega^2) \left\{ \frac{\dot{B}}{B} - \frac{\dot{f}_R}{f_R} - \frac{\dot{C}}{C} - \frac{\omega C'}{CB} + \frac{\omega f'_R}{Bf_R} \right\} \right], \quad (25)$$

while  $g^\dagger = g_{,\mu}S^\mu$ ,  $g^* = g_{,\mu}U^\mu$  and  $\mathcal{D}_0$ ,  $\mathcal{D}_1$  contain Palatini  $f(R)$  dark sector quantities and are given in Appendix.

The total amount of matter content within the spherical stellar interior can be found through the well-known Misner-Sharp mass formalism [38]. For metric (18), it is given by

$$m(t, r) = \frac{C}{2} (1 - g^{\gamma\delta} C_{,\gamma} C_{,\delta}) = \left( 1 + \dot{C}^2 - \frac{C'^2}{B^2} \right) \frac{C}{2}. \quad (26)$$



Now, we define an operator which is related to coordinate  $r$  derivation as

$$D_C = \frac{1}{C'} \frac{\partial}{\partial r}. \quad (27)$$

Equation (26) can be manipulated as

$$E \equiv \frac{C'}{B} = \left[ 1 + U^2 - \frac{2m(t, r)}{C} \right]^{1/2}, \quad (28)$$

where  $U$  denotes fluid velocity which for LTB geometry is found as  $U = \dot{C}$ . The  $f(R, T)$  field equations and Eqs.(26)-(28) give

$$\begin{aligned} D_C m = & \frac{\kappa}{2f_R(1-\omega^2)} \left[ \tilde{\mu} \left( 1 + \frac{\omega U}{E} \right) + \tilde{P}_r \omega \left( \omega + \frac{U}{E} \right) + \tilde{q} \left\{ 2\omega + (1 + \omega^2) \frac{U}{E} \right\} \right. \\ & \left. + (1 - \omega^2) \left( \mathcal{T}_{00} - \frac{U \mathcal{T}_{01}}{EB} \right) \right] C^2, \end{aligned} \quad (29)$$

while the variation of LTB matter content respecting time is

$$\begin{aligned} \dot{m} = & \frac{-\kappa}{2f_R(1-\omega^2)} \left[ \left\{ (\tilde{\mu} + \tilde{P}_r) \omega + \tilde{q}(1 + \omega^2) - \frac{\mathcal{T}_{01}}{B}(1 - \omega^2) \right\} E + \{ \tilde{\mu} \omega^2 \right. \\ & \left. + \tilde{P}_r + 2\tilde{q} \omega + \frac{\mathcal{T}_{11}}{B^2}(1 - \omega^2) \right\} U \right] C^2. \end{aligned} \quad (30)$$

The radial integration of Eq.(30) yields

$$\begin{aligned} m = & \frac{\kappa}{2} \int_0^C \frac{1}{f_R(1-\omega^2)} \left[ \tilde{\mu} \left( 1 + \frac{\omega U}{E} \right) + \tilde{P}_r \omega \left( \omega + \frac{U}{E} \right) + \tilde{q} \left\{ 2\omega + (1 + \omega^2) \right. \right. \\ & \left. \left. \times \frac{U}{E} \right\} + (1 - \omega^2) \left( \mathcal{T}_{00} - \frac{U \mathcal{T}_{01}}{EB} \right) \right] C^2 dC, \end{aligned} \quad (31)$$

which can be reinterpreted as

$$\begin{aligned} \frac{3m}{C^3} = & \frac{3\kappa}{2C^3} \int_0^C \frac{1}{f_R(1-\omega^2)} \left[ \tilde{\mu} \left( 1 + \frac{\omega U}{E} \right) + \tilde{P}_r \omega \left( \omega + \frac{U}{E} \right) + \tilde{q} \left\{ 2\omega + (1 + \omega^2) \right. \right. \\ & \left. \left. \times \frac{U}{E} \right\} + (1 - \omega^2) \left( \mathcal{T}_{00} - \frac{U \mathcal{T}_{01}}{EB} \right) \right] C^2 dC. \end{aligned} \quad (32)$$

After decomposing Weyl tensor into its electric and magnetic parts, we found that magnetic component turn out to be zero for out LTB spherical structure. However, its electric part is non-zero. This can be represented via  $U_\gamma$  and  $S_\gamma$  as

$$E_{\gamma\delta} = \mathcal{E} \left[ S_\gamma S_\delta - \frac{1}{3}(g_{\gamma\delta} + U_\gamma U_\delta) \right],$$

with

$$\mathcal{E} = \left\{ \frac{\ddot{C}}{C} + \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) \frac{\dot{C}}{C} - \frac{\ddot{B}}{B} \right\} - \left\{ \frac{C''}{C} - \left( \frac{C'}{C} + \frac{B'}{B} \right) \frac{C'}{C} \right\} \frac{1}{2B^2} - \frac{1}{2C^2}. \quad (33)$$

This Weyl scalar can be written alternatively via mass function and field equations as

$$\mathcal{E} = \frac{\kappa}{2f_R} \left( \tilde{\mu} - \tilde{P}_r + P_\perp + \mathcal{T}_{00} - \frac{\mathcal{T}_{11}}{B^2} + \frac{\mathcal{T}_{22}}{C^2} \right) - \frac{3m}{C^3}. \quad (34)$$

This expression would be very useful to calculate Ellis equation with tilted congruences in Palatini  $f(R)$  gravity.

Now, we are interested to calculate Palatini  $f(R)$  distributions of Ellis equations by following the procedure given by Ellis [39]. These would help us to find irregularity factors in the energy density of dissipative locally anisotropic matter content with tilted congruences. Using Eqs.(29), (30), (33), (34) and tilted  $f(R, T)$  field equations, these are formulated as follows

$$\begin{aligned} & \left[ \mathcal{E} - \frac{\kappa}{2f_R} \left\{ \tilde{\mu} - \tilde{P}_r + P_\perp + \mathcal{T}_{00} - \frac{\mathcal{T}_{11}}{B^2} + \frac{\mathcal{T}_{22}}{C^2} \right\} \right]_{,0} = \frac{3\dot{C}}{C(1-\omega^2)} \left[ -\mathcal{E} + \frac{\kappa}{2f_R} \right. \\ & \times \left\{ \tilde{\mu}(1+\omega^2) + P_\perp + \mathcal{T}_{00} + 2\tilde{q}\omega - \frac{\mathcal{T}_{11}}{B^2}\omega^2 + \frac{\mathcal{T}_{22}}{C^2} \right\} \left. \right] + \frac{3\kappa C'}{2BC(1-\omega^2)f_R} \\ & \times \left[ (\tilde{\mu} + \tilde{P}_r)\omega + \tilde{q}(1+\omega^2) - \frac{\mathcal{T}_{01}}{B}(1-\omega^2) \right], \end{aligned} \quad (35)$$

$$\begin{aligned} & \left[ \mathcal{E} - \frac{\kappa}{2f_R} \left\{ \tilde{\mu} - \tilde{P}_r + P_\perp + \mathcal{T}_{00} - \frac{\mathcal{T}_{11}}{B^2} + \frac{\mathcal{T}_{22}}{C^2} \right\} \right]' = -\frac{3C'}{C(1-\omega^2)} \left[ \mathcal{E} + \frac{\kappa}{2f_R} \right. \\ & \times \left\{ \tilde{P}_r(1+\omega^2) - P_\perp - \mathcal{T}_{00}\omega^2 + 2\tilde{q}\omega + \frac{\mathcal{T}_{11}}{B^2} - \frac{\mathcal{T}_{22}}{C^2} \right\} \left. \right] - \frac{3\kappa UC'}{2EC(1-\omega^2)f_R} \\ & \times \left[ (\tilde{\mu} + \tilde{P}_r)\omega + \tilde{q}(1+\omega^2) - \frac{\mathcal{T}_{01}}{B}(1-\omega^2) \right]. \end{aligned} \quad (36)$$

On considering  $f(R) = R$  in above equations, GR Ellis equations for tilted congruences can be found. However, Ellis equation calculated by Herrera *et al.* [40] can be recovered by taking  $\omega = 0$  along with  $f(R) = R$ .

## 4 Inhomogeneities in the Tilted LTB Spheres

In this section, we would find factors disturbing the energy density inhomogeneity of the tilted LTB system coupled with anisotropic dissipative relativistic matter. We would solve modified versions of Ellis equations that has related the Weyl tensor with fluid source variables. The study of inhomogeneity parameters occupy enticing importance in the complete description of stellar gravitational collapse. The initial homogeneously evolving system will only enter in the collapsing window once it experiences energy density irregularities.

(i) What factors are actually creating these changes over the surface of regular relativistic system?

(ii) Are dark sector terms affect these inhomogeneity factors?

(iii) Furthermore, is this study an observer dependant?

In order to answer these issues, we would like to carry out our analysis with the help of Ellis equations in Palatini  $f(R)$  gravity. We shall also check the influence of kinematical parameters in the modeling of inhomogeneous phases of collapsing stellar objects. Since, modified gravity may results cumbersome set of linear equations, therefore, we would like to perform our analysis by taking simple case of matter source and then we will increase their order of complexity. We consider following calculations under the context of current cosmological Ricci scalar constraint. We shall classify our investigations into couple of streams, i.e., radiating/dissipative and non-radiating/non-dissipative populations.

### 4.1 Non-Radiating Case

This section explores inhomogeneity factors of the adiabatic relativistic tilted matter sources with LTB geometry as gravitational field in Palatini  $f(R)$  gravity. This section constitutes various non-dissipative choices of matter fields such as dust, perfect and anisotropic galactic populations, respectively.

#### 4.1.1 Cloud of Non-interacting Particles

First we check the geodesic cloud of non-interacting adiabatic relativistic fragments. So, we consider all pressure gradients, radiation density as well as heat flux to be zero. For this subsection, Eqs.(35) boils down to

$$\left[ \mathcal{E} - \frac{\kappa}{2(1 - \delta^4 R^{-2})} \left\{ \mu - \frac{\delta^4}{R\kappa} \right\} \right]_{,0} = \frac{3\dot{C}}{C(1 - \omega^2)} \left[ -\mathcal{E} + \frac{\kappa}{2(1 - \delta^4 R^{-2})} \left\{ \mu \right. \right. \\ \left. \left. \times (1 + \omega^2) - \frac{\delta^4}{R\kappa} \right\} \right] + \frac{3\kappa\mu\omega C'}{2BC(1 - \omega^2)(1 - \delta^4 R^{-2})},$$

which after using first dynamical equation can be read as

$$(1 - \omega^2)\dot{\mathcal{E}} + \frac{3\dot{C}}{C}\mathcal{E} = \frac{\kappa}{2(1 - \delta^4 R^{-2})} \left[ \left\{ \frac{3C'}{C} \frac{\omega}{B} - \Theta(1 - \omega^2)^{3/2} + \frac{3\dot{C}}{C} \right. \right. \\ \left. \left. \times (1 + \omega^2 - \delta^4 \omega^2) \right\} \mu + (\omega^2 - 1)\mu' \right]. \quad (37)$$

It is well-known fact that the energy density of dust particles are regular once the systems impregnate null Weyl scalar. This will consequently implies zero value of radial derivative of energy density. Using this result, above equation provides the following value of expansion scalar

$$\Theta = \frac{1}{\sqrt[3]{1 - \omega^2}} \left[ \frac{3\dot{C}}{C}(\delta^4 \omega^2 - \omega^2 - 1) - \frac{3C'}{C} \frac{\omega}{B} \right]. \quad (38)$$

If irregular system wish to enter in the regular window, its matter content should need to attain above value of expansion scalar. Now, the second Ellis equation (36) provides

$$\left[ \mathcal{E} - \frac{\kappa}{2(1 - \delta^4 R^{-2})} \left\{ \mu + \frac{\delta^4}{R\kappa} \right\} \right]' = -\frac{3C'}{C(1 - \omega^2)} \left[ \mathcal{E} + \frac{\kappa}{2(1 - \delta^4 R^{-2})} \right. \\ \left. \times \left\{ \frac{\delta^4}{R\kappa} \omega^2 \right\} \right] - \frac{3\kappa U C' \mu \omega}{2EC(1 - \omega^2)(1 - \delta^4 R^{-2})},$$

which provides the inhomogeneity condition

$$\frac{C'}{C} = \frac{RU\kappa\mu}{3\omega\delta^4 E(1 - \omega^2)}. \quad (39)$$

Using above relation in the solution of above Ellis equation with Schwarzschild radius, i.e.,  $C = r$ , give

$$\Theta = 0.$$

This shows that homogeneous tilted dust particles with Palatini  $f(R)$  corrections should satisfy expansion-free condition. Under this condition, the system would experience two very interesting dynamical process.

(i) This condition produces two distinct boundaries (within the spherical object) in which external one differentiates the relativistic matter content from the exterior vacuum metric while the interior one distinguishes central Minkowskian core from the fluid gravitational source. Under zero expansion scalar, the matter content evolves without being compressed.

For instance, during expansion of spherical stellar gradient, the changes in its volume produce similar expansion in the external hypersurface counterbalancing similar internal surface expansion. Thus, zero expansion scalar initiates a specific form of system evolution in such which the inner most shell drags away from the central point resulting the outcome of vacuum core. Based on this concept, expansion-free matter populations could be effective for the voids explanation.

(ii) The collapsing expansion-free fluid upon approaching towards the central point experienced shear scalar blowup. The strong shearing effects cause obstruction in the appearance of apparent horizon, thereby supporting the existence of naked singularity (NS) [41]. Thus, in nature, NS and expansion-free condition are weaved together. For the deep understanding of NS appearance, Virbhadra *et al.* [42] developed general formalisms. Further, Virbhadra and Ellis [43] linked this outstanding NS phenomenon with gravitational lensing and presented some basic foreground results.

#### 4.1.2 Locally Isotropic Matter Populations

Here, we assumed that tilted observer has witnessed that LTB relativistic metric is designed due to gravitational field produced by ideal matter sources in Palatini  $f(R)$  gravity. Then, Eq.(35) yields

$$\left[ \mathcal{E} - \frac{\kappa}{2(1 - \delta^4 R^{-2})} \left\{ \mu + \frac{\delta^4}{R\kappa} \right\} \right]_{,0} = \frac{3\dot{C}}{C(1 - \omega^2)} \left[ -\mathcal{E} + \frac{\kappa}{2(1 - \delta^4 R^{-2})} \right. \\ \left. \times \left\{ P(1 + \omega^2) + (\mu + P) - \frac{\delta^4}{R\kappa} \omega^2 \right\} \right] + \frac{3\kappa C'(\mu + P)\omega}{2BC(1 - \omega^2)(1 - \delta^4 R^{-2})},$$

after using Eq.(22), above equation provides

$$\dot{\mathcal{E}} + \frac{3\mathcal{E}\dot{C}}{C(1 - \omega^2)} = \frac{\kappa}{2(1 - \delta^4 R^{-2})} \left[ \frac{3\dot{C}}{C(1 - \omega^2)} - \Theta\sqrt{1 - \omega^2} + \frac{3\omega C'}{BC(1 - \omega^2)} \right] \\ \times \left\{ \mu - \frac{\alpha(R^2 - \delta^4)}{2(1 + 2\alpha R)} - \frac{\delta^4}{R\kappa} \right\} - \frac{3\dot{C}\delta^4\omega^2}{2C(1 - \omega^2 R)(1 - \delta^4 R^{-2})} - \frac{\kappa\mu'}{2B(1 - \delta^4 R^{-2})}. \quad (40)$$

It is worthy to stress that for comoving system, we have considered corrections coming from  $f(R) = R + \alpha R^2$  model [44, 45], in which  $\alpha$  is a positive number. Equation (40), after using some relations between tilted and non-tilted congruences, provides the following constraint for the existence of regular energy density with Palatini  $f(R)$  background

$$\Theta = \frac{3\delta^4\dot{C}\omega^2}{\kappa C R(1 - \omega^2)^{5/2}} \left( \frac{3\dot{C}}{C} + \frac{3\omega C'}{BC} \right) \left\{ \mu - \frac{\alpha(R^2 - \delta^4)}{2(1 + 2\alpha R)} - \frac{\delta^4}{R\kappa} \right\}^{-1}. \quad (41)$$

It is evident from the above equation that isotropic LTB stellar model having Schwarzschild radius will be homogeneous only when the system impregnates vacuum core. Now, it follows from Eq.(36) that

$$\left[ \mathcal{E} - \frac{\kappa}{2(1 - \delta^4 R^{-2})} \left\{ \mu + \frac{\delta^4}{R\kappa} \right\} \right]' = -\frac{3C'}{C(1 - \omega^2)} \left[ \mathcal{E} + \frac{\kappa\omega^2}{2(1 - \delta^4 R^{-2})} \right. \\ \left. \times \left\{ P - \frac{\delta^4}{R\kappa} \right\} \right] - \frac{3\kappa UC'(\mu + P)\omega}{2EC(1 - \omega^2)(1 - \delta^4 R^{-2})},$$

which can be interpreted as

$$\mathcal{E} + \frac{3C'\mathcal{E}}{C(1 - \omega^2)} = \frac{-3\kappa C'}{2(1 - \omega^2)(1 - \delta^4 R^{-2})} \left[ P\omega^2 - \frac{\delta^4\omega^2}{\kappa R} + \frac{U\omega}{E}(\mu + P) \right] \\ + \frac{\kappa\mu'}{2(1 - \delta^4 R^{-2})}, \quad (42)$$

from which inhomogeneity factor is found as

$$\Psi \equiv -\frac{E}{U} \left( \omega - \frac{U}{E} \right) \left\{ \frac{\alpha(R^2 - \delta^4)}{2(1 + 2\alpha R)} + \frac{\delta^4}{R\kappa} \right\} - \frac{\delta^2\omega}{R\kappa E} - \mu. \quad (43)$$

This shows that when the system is in inhomogeneous phase, it should need to make null contributions of Weyl scalar and  $\Psi$ . The major portion of expression  $\Psi$  is controlled by the dark sector terms coming from Palatini  $f(R)$  gravity. Thus,  $f(R)$  terms tends to make hindrance for the same to leave homogeneous as well as inhomogeneous phases due to their non-attractive nature.

#### 4.1.3 Locally Anisotropic Gravitational Sources

For this case, we consider all dissipative terms to be zero in the first Palatini  $f(R)$  Ellis equation. Then, it becomes

$$\left[ \mathcal{E} - \frac{\kappa}{2(1 - \delta^4 R^{-2})} \left\{ \mu - P_r + P_\perp - \frac{\delta^4}{R\kappa} \right\} \right]_{,0} = \frac{3\dot{C}}{C(1 - \omega^2)} \left[ -\mathcal{E} + \frac{\kappa}{2(1 - \delta^4 R^{-2})} \right. \\ \left. \times \left\{ \mu(1 + \omega^2) + P_\perp - \frac{\delta^4\omega^2}{R\kappa} \right\} \right] + \frac{3\kappa(\mu + P_r)\omega C'}{2BC(1 - \omega^2)(1 - \delta^4 R^{-2})}.$$

Equation (22), after performing some mathematical exercise, give

$$\left\{ \mathcal{E} + \frac{\kappa \Pi}{2(1 - \delta^4 R^{-2})} \right\}_{,0} + \frac{3\dot{C}}{C(1 - \omega^2)} \left\{ \mathcal{E} + \frac{\kappa \Pi}{2(1 - \delta^4 R^{-2})} \right\} = \frac{\kappa\mu}{2(1 - \delta^4 R^{-2})}$$

$$\left[ \frac{1}{(1-\omega^2)} \left\{ \frac{3\dot{C}}{C}(2+\omega^2) - \frac{6\omega C'}{BC} \right\} - \Theta\sqrt{1-\omega^2} \right] - \frac{3\kappa}{2(1-\delta^4 R^{-2})(1-\omega^2)} \\ \left( \frac{\dot{C}}{C} - \frac{\omega C'}{BC} \right) \frac{\hat{\mu}(R^2 - \delta^4)}{R^2(1+2\alpha R)} + \frac{D_2}{2(1-\delta^4 R^{-2})}, \quad (44)$$

with

$$D_2 = \left\{ \frac{\alpha(R^2 - \delta^4)}{2(1+2\alpha R)} + \frac{\delta^4}{R} \right\} \left[ \Theta\sqrt{1-\omega^2} - \frac{6}{(1-\omega^2)} \left( \frac{\dot{C}}{C} - \frac{\omega C'}{BC} \right) \right].$$

It can be seen from the literature that the scalar quantity that controls the inhomogeneity emergence in anisotropic sources is the trace free part of the tensor that came from the orthogonal splitting of Riemann curvature tensor. Such scalar has been dubbed as  $X_{TF}$ . We found that the configurations of squiggly brackets terms in first and second mathematical expressions of the above equation resemble with  $X_{TF}$ . Using this result, we find that following value of expansion which the anisotropic stellar populations must attain to achieve homogeneity in their energy densities.

$$\Theta = \frac{3}{\sqrt[3]{1-\omega^2}} \left[ (2+\omega^2) \frac{\dot{C}}{C} - \frac{2\omega C'}{BC} - \frac{\kappa \hat{\mu}(R^2 - \delta^4)}{R^2(1+2\alpha R)} \left( \frac{\dot{C}}{C} - \frac{\omega C'}{BC} \right) + \frac{D_2}{3}(1-\omega^2) \right]. \quad (45)$$

The second Ellis equation for anisotropic sources boils down to

$$\left[ \mathcal{E} - \frac{\kappa}{2(1-\delta^4 R^{-2})} \left\{ \mu - P_r + P_\perp - \frac{\delta^4}{R\kappa} \right\} \right]' = -\frac{3C'}{C(1-\omega^2)} \left[ \mathcal{E} + \frac{\kappa}{2(1-\delta^4 R^{-2})} \right. \\ \left. \times \left\{ P_r(1+\omega^2) - P_\perp - \frac{\delta^4 \omega^2}{R\kappa} \right\} \right] - \frac{3\kappa(\mu + P_r)\omega U C'}{2EC(1-\omega^2)(1-\delta^4 R^{-2})}.$$

This equation after some lengthy but easy mathematical manipulations yields

$$\left\{ \mathcal{E} + \frac{\kappa \Pi}{2(1-\delta^4 R^{-2})} \right\}' + \frac{3C'}{C(1-\omega^2)} \left\{ \mathcal{E} + \frac{\kappa \Pi}{2(1-\delta^4 R^{-2})} \right\} = \frac{\kappa(\mu + D_1)}{2(1-\delta^4 R^{-2})} \\ + \frac{\kappa U \omega}{2E(1-\omega^2)(1-\delta^4 R^{-2})} \left\{ \mu + \frac{\hat{\mu}(R^2 - \delta^4)}{R^2(1+2\alpha R)} \right\}, \quad (46)$$

where

$$D_1 = \frac{\kappa}{2(1-\delta^4 R^{-2})(1-\omega^2)} \left[ \frac{3C'\omega^2\delta^4}{CR} + \frac{\omega}{E} \left\{ \frac{2\delta^4}{R\kappa} + \frac{\alpha(R^2 - \delta^4)}{R^2(1+2\alpha R)} \right\} \right].$$

Here, we also noted the same mathematical combinations as we observed in Eq.(44). Therefore, the regular energy density can be achieved by the system if the system makes null value to the following parameter,  $\Phi$

$$\Phi \equiv \frac{R^2(1 - 2\alpha R)}{R^2 - \delta^4} \left\{ \frac{D_1}{U\omega}(1 - \omega^2) + \mu \right\} - \hat{\mu}. \quad (47)$$

For Schwarzschild radius, Eq.(45) provides non-zero value to expansion scalar. In comparison with previous cases,  $\Theta = 0$  was the necessary condition for dust and isotropic fluid systems to attain homogeneous energy density.

## 4.2 Radiating Case

In this subsection, we explore irregularity factor for the tilted observer who observed that LTB geometry of the stellar object is formed due to dissipative dust source. This cloud is dissipating in the mode of both diffusion and free-streaming approximations. Therefore, we take all anisotropic pressure gradients to be zero, then Eqs.(35) and (36) give

$$\begin{aligned} \left[ \mathcal{E} - \frac{\kappa}{2(1 - \delta^4 R^{-2})} \left\{ \tilde{\mu} - \frac{\delta^4}{R\kappa} \right\} \right]_{,0} &= \frac{3\dot{C}}{C(1 - \omega^2)} \left[ -\mathcal{E} + \frac{\kappa}{2(1 - \delta^4 R^{-2})} \right. \\ &\times \left. \left\{ \tilde{\mu}(1 + \omega^2) + 2\tilde{q}\omega - \frac{\delta^4 \omega^2}{R\kappa} \right\} \right] + \frac{3\kappa[\tilde{\mu} + \tilde{q}(1 + \omega^2)]C'}{2BC(1 - \omega^2)(1 - \delta^4 R^{-2})}, \end{aligned} \quad (48)$$

$$\begin{aligned} \left[ \mathcal{E} - \frac{\kappa}{2(1 - \delta^4 R^{-2})} \left\{ \tilde{\mu} - \frac{\delta^4}{R\kappa} \right\} \right]' &= -\frac{3C'}{C(1 - \omega^2)} \left[ \mathcal{E} + \frac{\kappa}{2(1 - \delta^4 R^{-2})} \right. \\ &\times \left. \left\{ 2\tilde{q}\omega - \frac{\delta^4 \omega^2}{R\kappa} \right\} \right] - \frac{3\kappa U[\tilde{\mu}\omega + \tilde{q}(1 + \omega^2)]C'}{2EC(1 - \omega^2)(1 - \delta^4 R^{-2})}. \end{aligned} \quad (49)$$

The second of above equation, after making some lengthy calculations, provides

$$\begin{aligned} \mathcal{E}' + \frac{3C'}{C(1 - \omega^2)} \mathcal{E} &= \frac{\kappa\tilde{\mu}'}{2(1 - \delta^4 R^{-2})} - \frac{3C'}{C(1 - \omega^2)(1 - \delta^4 R^{-2})} \left[ \frac{\kappa\omega\tilde{\mu}U}{2E} - \frac{\omega^2\delta^4}{R} \right. \\ &\quad \left. + \kappa\tilde{q} \left\{ \omega + \frac{U(1 + \omega^2)}{2E} \right\} \right], \end{aligned} \quad (50)$$

from which, we have obtained the constraint on heat conducting scalar as follows

$$\tilde{q} = \frac{\omega}{2\kappa} \left( \frac{\kappa U\tilde{\mu}}{E} - \frac{\omega\delta^4}{R} \right) \left\{ \omega + \frac{U(1 + \omega^2)}{2E} \right\}^{-1}. \quad (51)$$

The dissipative dust with tilted congruences will be of regular energy density, if  $\mu'$ , Weyl scalar as well as above value of heat flux is zero. This clearly shows that homogeneity depends upon dark source Palatini  $f(R)$  terms and congruence radial velocity  $\omega$ .



## 5 Summary

We have seen that LTB spacetimes as seen by a tilted observer exhibit physical properties which drastically differ from those present in the standard non-tilted LTB.

In this paper, we have studied the dynamics of LTB anisotropic geometry from the point of view of a tilted observer in Palatini  $f(R)$  gravity. The non-ideal matter distribution and the congruence supported by its 4-velocity vector, as observed by the tilted observer, is analyzed in detail. The “inhomogeneity factor”, i.e., the variable quantities depicting those aspects of the matter configurations that are involved in the emergence of energy-density irregularities, has been explored with respect to the tilted congruence. We have also integrated evolution of such factor in the maintenance of homogeneous phases of compact objects.

The dynamical equations and kinematical quantities are explored in non-comoving coordinates for the systematic construction of our analysis. Two expressions widely known to be Ellis equations have been developed in the context of Palatini  $f(R)$  gravity. These equations have linked the Weyl tensor with the material variables as seen by the tilted observer. We have extracted the factors that are responsible for the emergence of inhomogeneities in the LTB energy density under particular cases of dissipative and non-dissipative regimes. In non-radiating sector, we studied irregularity factors for the cloud of non-interacting particles, isotropic fluid and anisotropic matter while the radiating sector is explored only for non-interacting particles in tilted frame. The results in these particular cases are summarized as follows.

- For non-interacting and non-dissipative particles, we observed that the initially homogeneous system will remain homogeneous if it is conformally flat or have zero expansion scalar. It means that the inhomogeneity in the LTB type universe is not only controlled by the Weyl tensor but also the expansion scalar. We would like to stress here that the converse is not true, i.e., zero expansion condition does not lead to a homogeneous density distribution. It is worth mentioning that expansion-free scenarios have their own physical interpretation during the evolutionary process with some crucial impact on realistic models that we mentioned earlier in subsection 4.1.1.
- With the inclusion of isotropic pressure in the non-interacting particles, we found that the effects on the inhomogeneity parameters of density distribution differs from the previous one. We observed that a geometrical combination of dark source terms of Palatini  $f(R)$  gravitational field  $\Psi$  along with Weyl tensor and expansion scalar are the responsible factors. In the absence of extra curvature invariants of the theory, the Weyl tensor will be the only candidate for the appearance of inhomogeneities in the density distribution. Furthermore, we have explored that if during evolution, the

expansion scalar is able to attain a specific value (Eq.(41)), then the system will have regular environment of energy density.

- Similar factors for the case of non-radiating anisotropic matter distribution are obtained. For the smooth distribution of energy density, the value of  $\Theta$  has also been identified (mentioned in Eq.(45)). The corresponding inhomogeneity factor  $\Phi$  has also been explored (Eq.(47)). It is seen that Palatini  $f(R)$  dark source terms and tilted parameter  $\omega$  have produced hindrances for the system to leave initial homogeneous state of the compact object.
- In the radiating dust cloud case, we observed that a specific value of dissipation obtained in Eq.(51) is the responsible factor of density inhomogeneity in Palatini  $f(R)$  gravity and tilted observer along with the Weyl tensor. The homogeneous state can be recovered if the system is conformally flat and non-radiating.

All of our results support the analysis of [23] on setting  $\omega = 0$ , while the assumptions  $\omega = 0$  and  $f(R) = R$ , in our calculations would provide results compatible with [20] and [35].

## Acknowledgment

This work was partially supported by the JSPS KAKENHI Grant Number JP 25800136 and the research-funds presented by Fukushima University (K.B.).

## Appendix

The parts of Eqs.(22) and (23) are

$$\begin{aligned}\mathcal{D}_0 &= -\dot{\mathcal{T}}_{00} + \left(\frac{\mathcal{T}_{10}}{B^2}\right)' + \frac{\mathcal{T}_{01}}{B^2} \left(\frac{2\dot{f}_R}{f_R} + \frac{B'}{B} + \frac{2C'}{C}\right) - \mathcal{T}_{00} \left(\frac{\dot{B}}{B} + \frac{3\dot{f}_R}{2f_R} + \frac{2\dot{C}}{C}\right) \\ &\quad - \frac{2\mathcal{T}_{22}}{C^2} \left(\frac{\dot{C}}{C} + \frac{\dot{f}_R}{2f_R}\right) - \frac{\mathcal{T}_{11}}{B^2} \left(\frac{\dot{B}}{B} + \frac{\dot{f}_R}{2f_R}\right), \\ \mathcal{D}_1 &= \mathcal{T}_{00} \frac{\dot{f}_R}{2f_R} - \dot{\mathcal{T}}_{10} + \left(\frac{\mathcal{T}_{11}}{B^2}\right)' - \mathcal{T}_{10} \left(\frac{2\dot{f}_R}{f_R} + \frac{\dot{B}}{B} + \frac{2\dot{C}}{C}\right) - \frac{2\mathcal{T}_{22}}{C^2} \left(\frac{C'}{C} + \frac{\dot{f}_R}{2f_R}\right)\end{aligned}$$

$$+ \frac{\tau_{11}}{B^2} \left( 2 \frac{C'}{C} + \frac{3f'_R}{2f_R} \right).$$

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